Example Problems-2 - probs 3, 4

Expectation of a binomial random variable with parameters p and n is pn. So, $E(Y|X=n) = \frac{n}{2}$ since $p = \frac{1}{2}$. Using law of total probability:
$E(Y) = \sum_{n=1}^{6} E(Y|X=n)P(X=n) = \frac{1}{6}\sum_{n=1}^{6}\frac{n}{2} = \frac{7}{4}$

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Similarly,

$$\begin{aligned}
E(XY) &= \sum_{n=1}^{6} E(XY|X=n)P(X=n) \\
&= \sum_{n=1}^{6} E(nY|X=n)P(X=n) \\
&= \frac{1}{6}\sum_{n=1}^{6} nE(Y|X=n) \\
&= \frac{1}{6}\sum_{n=1}^{6} n\frac{n}{2} = \frac{91}{12}
\end{aligned}$$

Example Problems-3. Three prisoners problem - conditional probabilities

Three prisoners A, B, C are to be executed. The queen announces she will pardon one of them. She randomly picks one and tells the jailer but asks her not to reveal the name.

A sees the jailer and asks her who has been pardoned. She refuses to tell. A then asks who of B or C will be executed. The jailer thinks a bit and says 'B is being executed'.

Has the jailer's remark changed the probability of A being pardoned?

Here is the reasoning of each party:

Jailer:The probability of any prisoner being pardoned is $\frac{1}{3}$. One of B or C must be executed so revealing that B is being executed did not change A's probability of being executed.

A: Since B will be executed only A or C can be pardoned so A's probability of being pardoned is $\frac{1}{2}$.

Who is right? I

Let A, B, C be the events that A, B, C respectively were pardoned.
Let J be the event jailer says 'B is being executed'.
We have to find $P(A|J) = \frac{P(A \cap J)}{P(J)}$
What happened is summarized below:

Pardoned	Jailer says
A	'B being executed' $\mid$ 'C being executed' - both equiprobable i.e. $\frac{1}{2}$
B	'C being executed'
C	'B being executed'

$$
\begin{aligned}
P(J) &= P(\text{jailer says 'B being executed'}) \\
&= P(J \cap \text{A pardoned}) + P(J \cap \text{C pardoned}) + P(J \cap \text{B pardoned}) \\
&= \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times 1 + \frac{1}{3} \times 0 \\
&= \frac{1}{2}
\end{aligned}
$$

Who is right? II

So, $P(A|J) = \frac{P(A \cap J)}{P(J)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

A wrongly interprets J as B^c giving $P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

Example Problems-4

- In a coin tossing game you pay your friend Rs.10/- if your call is wrong and he pays you Rs.10/- if your call is correct.
- You lose Rs.100/- in 10 successive tosses where you called Heads but the coin turned up tails.
- Assuming the coin is fair what should you call on the eleventh toss?

Example Problems-4

- In a coin tossing game you pay your friend Rs.10/- if your call is wrong and he pays you Rs.10/- if your call is correct.
- You lose Rs.100/- in 10 successive tosses where you called Heads but the coin turned up tails.
- Assuming the coin is fair what should you call on the eleventh toss?

Since the coin is fair you can call either Heads or Tails it does not matter.

Examples: 2

1. There are 3 identical boxes (say 1, 2, 3) two of which contain Rs.10/- and the third has Rs.1000/-. The choice of boxes for putting the amounts is random. Also, the experimenter is told which box contains what amount.
2. You, the player, choose a particular box.
3. The experimenter chooses a second box and shows that it contains Rs.10/-. And gives you the option to switch to the remaining box, if you wish.
4. Should you switch?

Examples: 2 contd.

Assume you choose Box 1. Let us analyse outcomes

Box 1	**Box 2**	**Box 3**	**Stay**	**Switch**
10	10	1000	10	1000
10	1000	10	10	1000
1000	10	10	1000	10

Examples: 2 contd.

Assume you choose Box 1. Let us analyse outcomes

Box 1	**Box 2**	**Box 3**	**Stay**	**Switch**
10	10	1000	10	1000
10	1000	10	10	1000
1000	10	10	1000	10

So, better to switch. Probability of winning Rs.1000/- is $\frac{2}{3}$ while staying has a probability of $\frac{1}{3}$.

Probability can play tricks with intuition

- Simpson's paradox. (Already discussed earlier)
- Gambler's fallacy.
- Monty Hall problem.

Examples: 4

Figure: The number of • is same on opposite sides

The game has two players You and I:
The players select one of the dice and roll. The winner is the one who rolls the higher number. Assume You select the die first.

Analysis of game - version 1

Analysis with (die A, die B):

$\mathcal{S}(S) = \{(2,1),(2,5),(2,9),(6,1),(6,5),(6,9),(7,1),(7,5),(7,9)\}$

The probability of each outcome is $\frac{1}{9}$. Probability Die A wins is $\frac{5}{9}$ and B wins is $\frac{4}{9}$. Repeating such an analysis with (die A, die C) and (die B, die C) gives the following table:

(die 1, die 2)	**Prob. die 1 wins**	**Prob. die 2 wins**
(A,B)	$^5/_9$	$^4/_9$
(A,C)	$^4/_9$	$^5/_9$
(B,C)	$^5/_9$	$^4/_9$

The player choosing first is always worse off since $A \succ B \succ C \succ A$ (where $\succ$ is better than or higher probability than relation). Note that the $\succ$ relation is not transitive.

Analysis of game - version 2

Analysis: each die A. B rolled twice and summed. Let A, B be the random variables for the sums for A, B respectively.

$B \in \{2, 6, 6, 10, 10, 10, 14, 14, 18\}$ and
$A \in \{4, 8, 8, 9, 9, 12, 13, 13, 14\}$. We have 81 possible pairs as the outcome of the game. A's win probability is $^{37}/_{81}$, B's is $^{42}/_{81}$ and there is a tie with probability $^{2}/_{81}$.

A similar analysis for the others yields the following table:

(die 1, die 2)	**Prob. die 1 wins**	**Prob. die 2 wins**	**Draw**
(A,B)	$^{37}/_{81}$	$^{42}/_{81}$	$^{2}/_{81}$
(A,C)	$^{44}/_{81}$	$^{33}/_{81}$	$^{4}/_{81}$
(B,C)	$^{37}/_{81}$	$^{42}/_{81}$	$^{2}/_{81}$

In this case we have: $A \prec B \prec C \prec A$. Choosing first is still bad but the relations have reversed.

Probabilities do not follow intuitive rules.

Law of large numbers

Theorem 1 (Weak law of large numbers)

Sample average converges in probability towards the expected value as sample size $n \to \infty$.

$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$

Theorem 2 (Strong law of large numbers)

Sample average converges almost surely to the expected value as sample size $n \to \infty$.

$P(\lim_{n\to\infty} \bar{X}_n = \mu) = 1$