Analysis of Variance - ANoVA

- The null hypothesis when we have different groups with different treatments is that the groups belong to the same population which implies the means of the groups are the same.
- The basic idea is to compare the variability due to the treatment or intervention with pure statistical variability between samples. This is the reason the analysis technique is called *Analysis of Variance*.

Example experiment - one independent variable two levels

How does mental imagery affect memory?
Two groups of subjects (say G1, G2) were asked to memorize pairs of words and then recall the pair given the first word as a cue after a 24 hours gap. G1 was instructed to actively use imagery to link the two words and given examples of such linking. G2 was just instructed to remember the pairs in whatever way they could. The number of pairs recalled was the response variable. The independent variable is the group (or treatment) and levels are G1-imagery, G2-control.
The results for 10 subjects with 5 subjects randomly assigned to each of G1, G2 are given below for 20 word pairs.

G1	**G2**
8	1
8	2
9	5
11	6
14	6

G1 mean=10; G2 mean=4; Grand mean=7.

Reasons for variability

- There are two possible reasons why the means of the different groups are different:
  - A possible effect of the treatment/intervention. This is the desired result.
  - Effect of sampling fluctuations, individual differences, other errors. This is the statistical or experimental error.

  Summarized as:

  $$\text{Between-group variability} = \text{effect of independent variable(s)} + \text{statistical/experimental error}$$

- Within each group the variability can only be due to individual differences, sampling fluctuation etc. since the treatment/ intervention is the same for indviduals in the group. So,

  $$\text{Within-group variability} = \text{statistical/experimental error}$$

- Using these two variabilities it is possible to construct an index of the effect of the independent variable(s).

Index of the effect - F-ratio. Informal

- One index of characteriziation that does not depend on the unit of measurement
  $F = \frac{\text{variability between groups}}{\text{variability within groups}} = \frac{\text{effect of indp. var(s)} + \text{error}}{\text{error}}$
- If there is no effect $F = \frac{\text{est. of error (between groups)}}{\text{est. of error (within groups)}}$
  Since the numerator and denominator is estimating the same quantity we expect F to be close to 1. If there is an effect then $F > 1$ and the larger the effect the larger the value of F.

ANoVA sum of squares breakup for single factor I

- Assume there is only one factor and let ℓ be the number of levels for the factor. This means there are ℓ groups. Let each group have equal number of subjects - N. So, $\ell \times N$ is the total number of subjects. Unequal number of subjects per group is handled later.
- Let μ_{grand} be the grand mean of the dependent variable Y.

$$\mu_{\text{grand}} = \frac{\sum_{i=1..\ell} \sum_{j=1..N} Y_{ij}}{\ell \times N}$$

Let μ_{l} be the group mean for level l: $\mu_{\text{l}} = \frac{\sum_{j=1..N} Y_{lj}}{N}$. Also, $\mu_{\text{grand}} = \frac{\sum_{l=1..\ell} \mu_{\text{l}}}{\ell}$

ANoVA sum of squares breakup for single factor II

The total sum of squares of deviation from the mean μ_{grand} is:

$$
\begin{aligned}
SS_{\text{total}} &= \sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{grand}})^2 \\
&= \sum_{l=1..\ell} \sum_{j=1..N} [(Y_{lj} - \mu_{\text{l}}) + (\mu_{\text{l}} - \mu_{\text{grand}})]^2 \\
&= \sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{l}})^2 + \sum_{l=1..\ell} \sum_{j=1..N} (\mu_{\text{l}} - \mu_{\text{grand}})^2 + \\
2 &\sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{l}})(\mu_{\text{l}} - \mu_{\text{grand}}) \\
&= \sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{l}})^2 + \sum_{l=1..\ell} \sum_{j=1..N} (\mu_{\text{l}} - \mu_{\text{grand}})^2 + 0
\end{aligned}
$$

The cross product term $\sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{l}})(\mu_{\text{l}} - \mu_{\text{grand}})$ is 0.

$$
= SS_{\text{withingroups}} + SS_{\text{betweengroups}}
$$

Variance like measures

- The sum-of-squares $SS_{\text{something}}$ depends on the number of elements. To compare different sums-of-squares they have to be normalized. This is done by defining: $s^2_{\text{something}} = \frac{SS_{\text{something}}}{df_{\text{something}}}$ where $df_{\text{something}}$ is the degrees of freedom.
- Define:

$$\begin{aligned} s^2_{\text{withingroups}} &= \frac{SS_{\text{withingroups}}}{df_{\text{withingroups}}} \\ s^2_{\text{betweengroups}} &= \frac{SS_{\text{betweengroups}}}{df_{\text{betweengroups}}} \end{aligned}$$

Degrees of freedom and variances I

- In statistics the **degree of freedom** of an estimate from a sample is the number of elements that can vary independently. For example, to calculate variance we need the mean. So, if sample size is N and the mean, say M, is known the number of elements that can vary independently is $N-1$ since $x_N = N \times M - \sum_{i=1..N-1} x_i$, only $N-1$ of the N elements can vary independently. So, the degree of freedom is $N-1$.
- To calculate the degrees of freedom of a sum we can use:

  df=number of observations - number of estimated parameters of the population.
- For $df_{\text{betweengroups}}$, $SS_{\text{betweengroups}} = \sum_{l=1..\ell}(\mu_{\text{l}} - \mu_{\text{grand}})^2$, μ_{grand} is needed and there are ℓ group means μ_{l} so $df_{\text{betweengroups}} = \ell - 1$.

Degrees of freedom and variances II

- For $df_{\text{withingroups}}$, $SS_{\text{withingroups}} = \sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{l}})^2$ there are $\ell \times N$ observations and ℓ means so $df_{\text{withingroups}} = N\ell - \ell = \ell(N-1)$.
- For df_{total}, $SS_{\text{total}} = \sum_{l=1..\ell} \sum_{j=1..N} (Y_{lj} - \mu_{\text{grand}})^2$ there are $\ell \times N$ observations and one estimated parameter μ_{grand} so $df_{\text{total}} = N\ell - 1$.
- Observe that

$$
\begin{aligned}
df_{\text{total}} &= N\ell - 1 = (N\ell - \ell) + (\ell - 1) \\
&= df_{\text{withingroups}} + df_{\text{betweengroups}}
\end{aligned}
$$

- In summary we get:

$$
\begin{aligned}
s^2_{\text{withingroups}} &= \frac{SS_{\text{withingroups}}}{df_{\text{withingroups}}} \\
s^2_{\text{betweengroups}} &= \frac{SS_{\text{betweengroups}}}{df_{\text{betweengroups}}} \\
s^2_{\text{total}} &= \frac{SS_{\text{total}}}{df_{\text{total}}}
\end{aligned}
$$

Degrees of freedom and variances III

- While sum-of-squares and degrees of freedom add up the variances do not. So, $s^2_{\text{total}} \neq s^2_{\text{withingroups}} + s^2_{\text{betweengroups}}$

Fisher F ratio

- The F ratio is defined by:

$$F = \frac{s^2_{\text{betweengroups}}}{s^2_{\text{withingroups}}}$$

- The null hypothesis H_0 is: that all groups belong to the same population so there is no difference in the group means. That is $\mu_1 = \mu_2 = \ldots = \mu_\ell$.
- The sampling distribution of the F-ratio is the Fisher F distribution and the critical values are obtained using $\nu_1 = df_{\text{betweengroups}}$ and $\nu_2 = df_{\text{withingroups}}$ and the chosen α level.
- H_0 is rejected if $F \geq F_{\text{critical}}$.

F Distribution: PDF

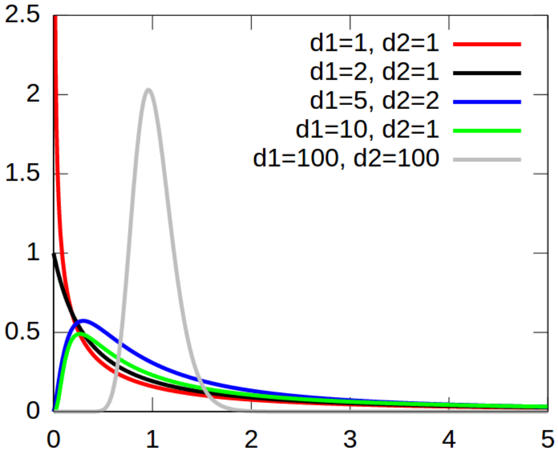

Figure: The PDF of the F distribution for different $\nu_1 = d_1$ and $\nu_2 = d_2$.
Source: Wikipedia

F Distribution: CDF

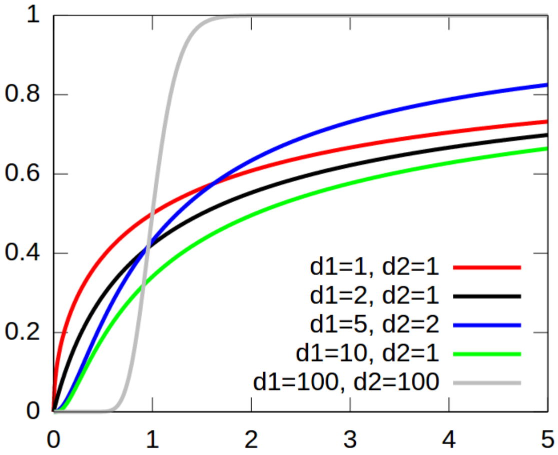

Figure: The CDF of the F distribution for different $\nu_1 = d_1$ and $\nu_2 = d_2$.
Source: Wikipedia

Exercise: ANoVA one factor, two levels

- One factor, two levels implies two groups. Assume each group has the same number of subjects N.
- The example imagery experiment fits this pattern.

Exercise 1

For the data in the imagery example calculate the following:

1. *The values of* $SS_{betweengroups}$, $SS_{withingroups}$, SS_{total}.
2. $df_{betweengroups}$, $df_{withingroups}$, df_{total}.
3. $s^2_{betweengroups}$, $s^2_{withingroups}$, s^2_{total}. *Verify that* $s^2_{total} \neq s^2_{betweengroups} + s^2_{withingroups}$.
4. *Compute the F-ratio.*
5. *Compute the critical values using* $\alpha = 0.05$ *and* $\alpha = 0.01$.
6. *Decide whether or not* H_0 *can be rejected at the two* α *levels.*
7. *What is the equivalent P-value?*